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LETTER TO THE EDITOR

The spectrum of fluctuations around Sompolinsky's mean field solution for a spin glass

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Abstract. The spectrum of the stability matrix associated with Sompolinsky's solution for a long-range spin glass is studied near T_c in a magnetic field. It is shown that the reparametrisation (or gauge) invariance of the theory locks together not only the order parameters $q(x)$ and $\Delta(x)$ but also their fluctuations, and gives rise to a gauge-invariant spectrum which is therefore the same both for the Parisi and for the Sompolinsky solution. In the limit of zero magnetic field our earlier results for the Parisi solution are recovered. The marginal stability of both theories is demonstrated for all fields in the Almeida–Thouless phase.

One of the most fascinating features of the recently proposed mean field theories of spin glasses is the appearance of a continuum of order parameters: the replica symmetry-breaking solution by Parisi (1979) of the model of Sherrington and Kirkpatrick (1975, to be referred to as SK) is based on an order parameter (OP) function $q(x)$ defined over the interval $0 \leqslant x \leqslant 1$, while by a completely different, dynamic approach Sompolinsky (1981) was led to a free energy functional F depending upon, besides the OP function $q(x)$, an anomaly function $\Delta(x)$. The requirement of stationarity of F can be shown to provide a relationship between q and Δ. This in turn allows the anomaly to be eliminated, leading back to a single OP function theory. Indeed, there is ample evidence now (though complete proof is still lacking‡) for the equivalence of the two theories as to their thermodynamic consequences. Thermodynamic equivalence on the mean field level does not necessarily imply full equivalence for all possible observables, however. Sompolinsky argued that 'once fluctuations are introduced there is no reason why independent variations in $q(x)$ and $\Delta(x)$ should not be considered'. This would then suggest that the eigenvalue spectrum of the stability matrix (Hessian) may be different in the two theories. One may wonder, for example, whether a zero mode exists also in Sompolinsky's theory. In fact, it transpired very clearly both in the restricted stability analysis of the Parisi solution by Thouless *et al* (1980, to be referred to as TAK) and in the more recent, complete analysis by the present authors (De Dominicis and Kondor 1982)§ that the existence of a zero eigenvalue of the Hessian was closely related to a particular feature of Parisi's $q(x)$,

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‡ Proof is still lacking of the monotonic behaviour of $\Delta'(x)/q'(x)$.
§ The stability analysis around the Parisi solution has also been performed independently by Goltsev (1982). His method exhibits eigenvalues of which only the third family agrees with ours, but does not provide eigenvector equations.

namely to the fact that it becomes independent of x beyond a certain interval (x_0, x_1) whose limits are determined by the temperature T and external field H. A flat piece in $q(x)$ gives no contribution to Sompolinsky's functional integral, however, so it can be regarded as unphysical, which then sheds doubt on the physical reality of the zero mode itself.

Motivated by the above considerations, we performed a stability analysis of the Sompolinsky solution, analogous to our previous work on Parisi's solution, but also including an external field this time. Our purpose here is to report the results of this work and to sketch how they can be obtained. Our principal conclusion is that the spectra of fluctuations around the two solutions are identical.

We take as a starting point the same truncated model free energy, introduced by Parisi (1979), that we used previously:

$$F\{q_{\alpha\beta}\} = -\lim_{n=0} \frac{1}{2n}\left(\tau \operatorname{Tr} q^2 + \tfrac{1}{3}\operatorname{Tr} q^3 + \tfrac{1}{6}\sum_{\alpha\neq\beta} q^4_{\alpha\beta} + H^2 \sum_{\alpha\neq\beta} q_{\alpha\beta}\right) \tag{1}$$

where $q_{\alpha\beta}$ is a real, symmetric $n\times n$ matrix, zero along the diagonal, n is the number of replicas and $\tau = (T_c - T)/T_c$ is the reduced temperature. Equation (1) is a good representation of the SK model near T_c. De Dominicis *et al* (1981) demonstrated that Sompolinsky's free energy functional can be derived on a purely static basis by considering a particular type of replica symmetry-breaking scheme. In that scheme one divides the OP matrix $q_{\alpha\beta}$ into big blocks of size $p_0\times p_0$ and makes a Parisi type of hierarchical subdivision of both the diagonal and off-diagonal big blocks into blocks of size p_k $(k = 1, 2, \ldots, R)$ with matrix elements q_k and r_k in the diagonal and off-diagonal blocks, respectively. In the limit when the replica number $n\to 0$ the p_k are able to tend to infinity in a prescribed order, namely such that $p_{k+1}/p_k \to 0$, while $p_k(q_k - r_k) = -\Delta'_k$ is kept finite. In the continuous limit $R\to\infty$, Δ' turns out to be of $O(1/R)$. For the free energy functional (1) then one finds

$$\begin{aligned} F = \tau\int_0^1 dt\, q(t)\Delta'(t) + \tfrac{1}{2}\tau q^2(1) - \tfrac{1}{3}q^3(1) - \int_0^1 dt\, q(t)\Delta'(t)\int_t^1 ds\,\Delta'(s) \\ - q(1)\int_0^1 dt\, q(t)\Delta'(t) + \tfrac{1}{3}\int_0^1 dt\, q^3(t)\Delta'(t) + \tfrac{1}{12}q^4(1) + \tfrac{1}{2}H^2\int_0^1 dt\,\Delta'(t). \end{aligned} \tag{2}$$

The functions $q(x)$ and $\Delta(x)$ are identified with Sompolinsky's OP functions; the prime on Δ means derivative. The stationarity conditions for (2) are

$$\Delta'(x)\left(\tau - \int_x^1 dt\,\Delta'(t) + q^2(x) - q(1)\right) \equiv \Delta'(x)G(x) = 0 \tag{3}$$

$$\tau q(x) - q(x)\int_x^1 dt\,\Delta'(t) - \int_0^x dt\, q(t)\Delta'(t) - q(1)q(x) + \tfrac{1}{3}q^3(x) + \tfrac{1}{2}H^2 = 0. \tag{4}$$

The x derivative of (4) gives $q'(x)G(x) = 0$; hence the two stationarity conditions are basically the same. (This is a general feature, unrelated to the simplification involved in the model (1), and it can be shown that the stationarity conditions always reduce to a form like $q'(x)G(x) = \Delta'(x)G(x) = 0$.) Returning now to conditions (3) and (4), we can see that if q' and Δ' vanish for some $x\in(0, 1)$ at all, they vanish simultaneously. ($q' = \Delta' = 0$ for all $x\in(0, 1)$ is the case of no symmetry breaking which we exclude here.) Whenever q', $\Delta' \neq 0$ they are connected by $G(x)=0$ leading, upon differentiation, to

$$\Delta'(x) = -2q(x)q'(x). \tag{5}$$

Using this and taking $G(x)=0$ at $x=1$ gives

$$\tau - q(1) + q^2(1) = 0 \tag{6}$$

which is the same as the condition for the maximum of Parisi's $q(x)$, while (4) taken at $x=0$ gives

$$q(0) = (\tfrac{3}{4}H^2)^{1/3}, \tag{7}$$

the minimum of Parisi's $q(x)$. No other conditions are imposed upon $q(x)$ and $\Delta(x)$, a fact reflecting the invariance of the theory with respect to a reparametrisation of x, analogous to a local gauge transformation, as stressed by Sompolinsky. For this reason we call $G(x)=0$, or its corollary (5), the gauge condition and note that it holds, in the ordered phase, for all $x \in (0, 1)$, irrespective of whether q' and Δ' vanish there or not. It is also clear that (3) and (4) do not select a single point in replica space as the solution of the extremum problem, but rather define an invariant orbit whose points should all be equivalent from the point of view of observables. Parisi's scheme corresponds to choosing a particular gauge, in which $q(x)=x/2$, $x_0<x<x_1$, with $x_0=2q(0)$, $x_1=2q(1)$, and $q(x)=$ constant otherwise. In order to be able to decide whether the existence of the zero mode depends on these flat pieces or not, we will, in contrast, commit ourselves to the type of solutions for which q' and Δ' do not vanish for any $0<x<1$ in the spin glass phase. For a sufficiently strong field, however, $q(0)$ becomes equal to $q(1)$, and one is forced to choose the solution $q' \equiv \Delta' \equiv 0$ of the stationarity equations, corresponding to the transition of de Almeida and Thouless (1978, to be referred to as AT).

Coming to the question of stability we can follow our earlier analysis for the Parisi scheme rather closely. It can, in particular, be seen that the simplest vectors solving the eigenvalue equations

$$(\lambda + 2\tau + 2q^2_{\alpha\beta})f_{\alpha\beta} + \sum_{\gamma \neq \alpha,\beta} (q_{\alpha\gamma}f_{\gamma\beta} + q_{\beta\gamma}f_{\gamma\alpha}) = 0 \qquad \alpha, \beta, \gamma = 1, 2, \ldots, n \tag{8}$$

of the Hessian associated with (1) have the same structure as the stationarity point itself, i.e. they can be represented by symmetric $n \times n$ matrices, zero along the diagonal, and subdivided into smaller and smaller blocks as described in the case of $q_{\alpha\beta}$ above. The matrix elements on the kth level of hierarchy will be called f_k and g_k in the diagonal and off-diagonal big blocks, respectively. The set of trial vectors of this structure constitutes our first family of eigenvectors.

Substituting this *ansatz* into (8) yields a set of coupled linear equations for f_k and g_k. In the limit $n=0$, $p_k \to \infty$, $p_{k+1}/p_k \to 0$ one observes again that the combination $h_k = -\lim_{p_k\to\infty} p_k(f_k - g_k)$ remains finite. Taking finally the continuous limit $R \to \infty$ one ends up with a set of two coupled integral equations for $f(x)$ and $h(x)$:

$$\lambda f(x) = 2q(x)\left(\int_x^1 \mathrm{d}t\, h(t) + f(1)\right) + 2\int_0^x \mathrm{d}t\, (q(t)h(t) + f(t)\Delta'(t)) \tag{9}$$

$$\lambda h(x) = 2\Delta'(x)\left(-2q(x)f(x) + f(1) + \int_x^1 \mathrm{d}t\, h(t)\right). \tag{10}$$

Taking the derivative of (9) and assuming $\lambda \neq 0$ we immediately recognise that a relation similar to (5) exists between h and f':

$$h(x) = -2q(x)f'(x). \tag{11}$$

Substituting (11) back into (9) reduces the problem to solving a single integral equation. In order to exhibit the gauge independence of the eigenvalues we go over into a manifestly invariant form by inverting the function $q(x)$ (permitted by $q' \neq 0$) and introducing $\varphi(q) = f(x(q))$. The stability equation then becomes

$$\tfrac{1}{2}\lambda\varphi(q) = q\varphi(q_1)(1-2q_1) + 2q\int_q^{q_1} \mathrm{d}t\,\varphi(t) + 2\int_{q_0}^{q} \mathrm{d}t\, t\varphi(t) + 2q_0^2\varphi(q_0) \tag{12}$$

where q_1 and q_0 are the extremal values of the independent variable q, given by (6) and (7) respectively. Repeated differentiation reduces (12) to the oscillator equation. The eigenvalues $\lambda \equiv 1/\omega^2$ are determined by

$$\cot 2\omega(q_1 - q_0) = \frac{\omega(1 - 2q_1 + 2q_0 + 8\omega^2 q_0^2 q_1)}{1 - 2\omega^2(q_0 - 2q_0q_1 + 2q_0^2)}. \tag{13}$$

This yields exactly the same spectrum as that found by TAK (de Almeida 1980) in the stability analysis of the Parisi solution. The spectrum is discrete and consists of a large eigenvalue $\lambda_0 \simeq 2q_1$, basically independent of q_0, and a series of small eigenvalues

$$\lambda_m = 4q_1^2/m^2\pi^2 \qquad m = 1, 2, \ldots \quad \text{for} \quad q_0 \approx 0$$

$$\lambda_m = \begin{cases} 8q_1^2(q_1 - q_0) & m = 1 \\ \dfrac{4(q_1-q_0)^2}{(m-1)^2\pi^2} & m = 2, 3, \ldots \end{cases} \qquad \text{for } q_0 \approx q_1. \tag{14}$$

All these eigenvalues are positive, but they accumulate around zero. On approaching the AT line all the small eigenvalues vanish.

At this point we have to remember that the possibility of finding a zero eigenvalue has been excluded when deriving the condition (11). In order to decide whether $\lambda = 0$ is admissible or not we go back to the original set of equations (9) and (10). Setting $\lambda = 0$ there we discover that the relation to replace (11) is now

$$h(x) = -2(q(x)f(x))' \tag{15}$$

and that the corresponding eigenvector is largely arbitrary, subject only to the condition $f(0) = f(1) = 0$.

It seems worthwhile emphasising that the set of coupled integral equations (9) and (10) could have been derived without making any reference to the replica method, but taking the derivatives of the Sompolinsky functional near T_c with respect to $q(x)$ and $\Delta'(x)$ directly. Therefore the spectrum found above cannot be regarded as an artefact of replicas; instead one has to conclude that the full TAK spectrum, including the zero mode, is there also in Sompolinsky's theory, despite it being free of the flat piece in $q(x)$.

The first family type eigenvectors considered so far do not exhaust all the possible solutions of the eigenvalue equation (8). In analogy with de Almeida and Thouless' (1978) stability analysis around the replica symmetric solution or with our own analysis around the Parisi solution, we also have to consider 'transverse' fluctuations corresponding to additional replica symmetry breaking. The second family of eigenvectors can then be constructed by selecting a distinguished replica in each of the big blocks $1, 2, \ldots, n/p_0$, and allowing the matrix elements $f_{\alpha\beta}$ to depend on the hierarchical distance to the distinguished replica in addition to depending on the hierarchical distance to the diagonal of the big block. (The precise definition of this new variable has been given by De Dominicis and Kondor (1982).) In the continuous limit we have then a pair of functions $f_z(x)$ and $h_z(x)$ defined over the unit square $0 \leqslant x, z \leqslant 1$. For $x \geqslant z$ these satisfy a set of equations identical to (9) and (10) with the new variable z playing the role of an added label only. For $x < z$ we find

$$\left(\lambda + \int_x^z \mathrm{d}t\, \Delta'(t)\right) f_z(x)$$
$$= \int_x^z \mathrm{d}t\, f_t(x)\Delta'(t) + q(x)\left(\int_x^1 \mathrm{d}t(h_x(t)+h_z(t)) + f_x(1) + f_z(1)\right)$$
$$+ \int_0^x \mathrm{d}t[\Delta'(t)(f_x(t)+f_z(t)) + q(t)(h_x(t)+h_z(t))] \qquad (16)$$

$$\left(\lambda + \int_x^z \mathrm{d}t\, \Delta'(t)\right) h_z(x)$$
$$= \Delta'(x)\left(-4q(x)f_z(x) + \int_x^1 \mathrm{d}t(h_x(t)+h_z(t)) + f_x(1) + f_z(1)\right)$$
$$+ \int_x^z \mathrm{d}t\, \Delta'(t)h_t(x). \qquad (17)$$

The complete set of these four coupled integral equations can then be solved by observing that (i) for $\lambda \neq 0$, h and f are again linked by a relation analogous to (11): $h_z(x) = -2q(x)\partial f_z(x)/\partial x$, $0 \leqslant z \leqslant 1$, hence h can be eliminated, and (ii) the solutions of the remaining two equations for $f_z(x)$, $x \gtreqless z$, can be classified by a continuous parameter $0 \leqslant \kappa \leqslant 1$ which plays the role of a breakpoint in the variable $z: f_z(x) \equiv f_\kappa(x)$, $z \geqslant \kappa$.

The eigenvalues in the second family then depend on κ, i.e. they form continuous bands. The $\kappa = 0$ edge of the bands coincides with the discrete spectrum found in the first family, which should be obvious from the fact that for $\kappa = 0$, $f_z(x)$ does not depend on z at all and (16) and (17) are then reduced to (9) and (10).

The whole second family spectrum can again be shown to be gauge invariant. If $q \equiv q(\kappa)$ the second family eigenvalues can be obtained from

$$\cot 2\omega(q_1 - q) = \frac{\omega(1-2q_1) + 2(q^2\omega^2-1)^{1/2}C(\xi)/C'(\xi)}{1 - 2\omega(1-2q_1)(q^2\omega^2-1)^{1/2}C(\xi)/C'(\xi)} \qquad (18)$$

where

$$\xi = q\omega/(q^2\omega^2-1)^{1/2} \qquad q_0 < q < q_1$$

and $C(z)$ is the solution of the Gegenbauer equation

$$(z^2-1)C''(z) + 4zC'(z) + 4C(z) = 0$$

belonging to the initial conditions

$$C(\xi_0) = \xi_0(1-\xi_0^2)/(1-3\xi_0^2) \qquad C'(\xi_0) = 1$$

with

$$\xi_0 = q_0\omega/(q^2\omega^2 - 1)^{1/2}. \tag{19}$$

In (18) and (19) the cut of the square root is along the negative real axis. In zero field ($q_0 = 0$) these equations go over into what we found earlier in the Parisi case: there is again a large eigenvalue $\lambda \approx 2q_1$, to leading order independent of q, while the small eigenvalues start from their TAK values for $q = 0$ ($\kappa = 0$), grow with increasing q and accumulate in a narrow strip around τ^2 for $q = q_1$. Approaching the AT line, $q_0 = q_1$, all the solutions disappear except the large eigenvalue which is basically independent of q_0 also.

An unstable mode ($\lambda < 0$) would correspond to a purely imaginary ω: it can be shown that no such solution of (18) and (19) exists.

The third and last family of eigenvectors can be constructed by distinguishing two replicas in each big block. The matrix elements $f_{\alpha\beta}$ will then depend on the hierarchical distance to the diagonal (x), the two distances to the two distinguished replicas (z_1, z_2), and finally on an additional, fixed parameter (ρ) specifying the distance between the two distinguished replicas. In the continuous limit we have a pair of functions $f^{\rho}_{z_1z_2}(x)$, $g^{\rho}_{z_1z_2}(x)$, $0 < x, z_1, z_2, \rho < 1$, which satisfy a set of 12 coupled integral equations. A relation analogous to (11) reduces the number of equations to six and the solutions can be characterised by a pair of breakpoints κ_1 and κ_2 in the two directions z_1 and z_2. The third family includes all the second family (just as the second included the first); the new eigenvalues correspond to the range in parameter space where $\kappa_1, \kappa_2 \geqslant \rho$. Here we can find the eigenvalues explicitly:

$$\lambda = p^2 + r^2 - 2q^2 \qquad q_0 \leqslant q \leqslant q_1 \quad q_1 \geqslant p, r \geqslant q \tag{20}$$

where p, r and q correspond to the values of the OP function $q(x)$ at κ_1, κ_2 and ρ, respectively. The eigenvalues in (20) fill the interval $(0, 2(q_1^2 - q_0^2))$; for zero field this is the same as what we found earlier in the Parisi case.

With this we have shown that the spectra of fluctuations around the Parisi solution and the Sompolinsky solution *as rederived via replicas* are the same. We have also pointed out that the first family eigenvalues could have been obtained without the use of replicas. As for the second and third family eigenvalues, their derivation, at least in the present form, is obviously connected to a particular replica symmetry-breaking scheme. One may ask if they have any relevance for Sompolinsky's theory in its *original, dynamic form.* The answer is that the eigenvalues of the Hessian have to show up as poles of various correlation functions. Sompolinsky and Zippelius (1983, and private communication) have recently made progress in calculating the correlation functions within the framework of their dynamic theory. They have been able in particular to determine the pole of four-point functions (double responses) in the limit when all three time intervals on which it depends are macroscopic, but one of them, t_0, separating the others, t_1 and t_2, is much larger (of the order of the largest relaxation time τ_0) than either t_1 or t_2 (of the order of the smallest relaxation time τ_1). If we regard our variables κ_1, κ_2 and ρ as the labels associated with the macroscopic times t_1, t_2 and t_0 according to Sompolinsky's interpretation of the argument of Parisi's $q(x)$, then the limit $t_0 \gg t_1, t_2$ translates into $\kappa_1, \kappa_2 > \rho \to 0$. We find it very rewarding that the pole found by Sompolinsky and Zippelius agrees with the $\rho = 0$ limit of our equation (20): $\lambda = p^2 + r^2$. It suggests that to recover the full equation (20) one should consider the case where t_0 becomes larger than but comparable with t_1 and t_2 in magnitude, while to find the second family eigenvalues (corresponding to $\kappa_1 = \kappa_2 < \rho$)

one should try to consider the reverse limit $t_0 < t_1, t_2$ or even $t_0 \ll t_1, t_2$ ($\kappa_1 = \kappa_2 = 0$, $\rho = 1$) for the first family. Calculating the poles of four-point functions may turn out to be rather hard in these cases, however.

To conclude, we remark that the key to the observed gauge invariance of the spectrum for $\lambda > 0$ is the relation (11) and its counterparts in the second and third families. The meaning of these relations is that not only q and Δ are linked at stationarity by a gauge condition, but also their fluctuations† around the extremum point. Replacing (11) by (15) (or by its analogues in the other two families) leads to a zero mode under very mild conditions being imposed upon the eigenvector, which seems to indicate that the zero mode carries an appreciable spectral weight. Approaching the AT line this weight must increase; right at the transition the spectrum consists of a large mass and zero only, thus smoothly joining the spectrum on the other side of the transition.

Details of this work will be published elsewhere.

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† Fluctuations are not independent above T_c either. This removes the paradox, pointed out to us by H Sompolinsky, which derives from the obviously non-definite quadratic form associated with fluctuations above T_c.